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STUDY OF RELATIVE CONNECTEDNESS AND QUASI NORMAL REGULARITY IN BITOPOLOGICAL SPACES.

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ABSTRACT

Our main aim is to derive the topological connectedness of strongly regular (WS-superregular, superregular) topological space (Y, τ') in X if and only if for each point $x \in Y$ and any neighborhood $U'(x) \in \tau'$ (respectively, $U(x) \in \tau$) there is a neighborhood $V(x) \in \tau$ (respectively, $V'(x) \in \tau'$, $V(x) \in \tau$) such that τ cl$V(x) \cap Y \subset U'(x)$ (respectively, τ cl$V'(x) \subset U(x)$, τ cl$V(x) \subset U(x)$).

Keywords: - topological, bitopological, compactness, connectedness normal, topological subspace, WS-supernormal,

Introduction

The bitopological properties and their topological counterparts are related with their respective counterparts which are clarified. We use the notations are (i) : TS for a topological space, (ii) TsS for a topological subspace, (iii) BS for a bitopological space and BsS for a bitopological subspace. $I, j \in \{1, 2\}$, $I \neq j$. We discuss the properties of relative separation axioms and relative connectedness, in particular, some relative versions of p-T_0, p-T_1, p-T_2, (I, j)- and p- regularities, (I,j)- and p-complete regularities, p-real normality and p-normality. Moreover, relative properties of (I, j)- and p- compactness types, including relative versions of (I, j)- and p-para-compactness, $(I, j)$and p-Lindelofness, (I, j)- and p-pseudo compactness are of great importance.

Theorem

Let (Y, τ_1', τ_2') *be a* BsS *of a* BS (X, τ_1, τ_2). Then *i* Y *is* (i, j)*-strongly regular in* X *if and only if for each point* $x \in Y$ *and any neighbourhood* $U'(x) \tau'_2$ *there is a neighborhood* $V(x) \in \tau_i$ *such that* $\tau_j \, \mathrm{cl}V(x) \cap Y \subset U'(x)$.

ii Y *is* (i, j)-WS-*superregular in* X *if and only if for each point* $x \in Y$ *and each neighborhood* $U(x) \in \tau_i$ *there is a neighbourhood* $V'(x) \in \tau'_i$ *such that* $\tau_j \, \mathrm{cl}V'(x) \subset U(x)$.

iii *is* (i, j)*-superregular in* X *if and only if for each point* $x \in Y$ *and each neighborhood* $U(x) \in \tau_i$ *there is a neighborhood* $V(x) \in \tau_i$ *such that* $\tau_j \, \mathrm{cl}V(x) \subset U(x)$.

Proof

we first prove (i) Let us consider, Y *to* be (i, j)-strongly regular in X, $x \in Y$ and $U'(x) \in \tau'_i$. Then $x \,\overline{\in}\, A = Y \setminus U'(x)$, $A \in \mathrm{co}\,\tau'_i$. Hence, there are $V(x) \in \tau_i$, $V(A) \in \tau_j$ such that $\tau_j \, \mathrm{cl}V(x) \cap V(A) = \emptyset$,

$\tau_j \, \mathrm{cl}V(x) \subset X \setminus V(A) \subset X \setminus A = X \setminus (Y \setminus U'(x)) = U'(x) \cup (X \setminus Y) \ldots$

and thus $\tau_j \, \mathrm{cl}V(x) \cap Y \subset U'(x)$.

Conversely, if the condition is satisfied, $x \in Y$, $A \in \mathrm{co}\tau_i'$ and $x \,\overline{\in}\, A$. Then $x \in Y \setminus A = U'(x) \in \tau_i'$ and hence, there is $V(x) \in \tau_i$ such that $\tau_j \, \mathrm{cl}V(x) \cap Y \subset U'(x)$. Therefore, we obtain the following relations.

$X \setminus U'(x) \subset X \setminus (\tau_j \, \mathrm{cl}V(x) \cap Y) = (X \setminus \tau_j \, \mathrm{cl}V(x)) \cup (X \setminus Y)$ so that $(X \setminus U'(x)) \cap Y \subset X \setminus \tau_j \, \mathrm{cl}V(x)$. Thus $A \subset X \setminus \tau_j \, \mathrm{cl}V(x) = V(A) \in \tau_j$ and $V(x) \cap V(A) = \emptyset$.

We now prove (ii) Let be (i, j)-WS-superregular in X, $x \in Y$ and $U(x) \in \tau_i$. Then $x \,\overline{\in}\, F = X \setminus U(x) \in \mathrm{co}\tau_i$ and so, there are $V(x) \in \tau_i'$, $V(F) \in \tau_j$ such that $\tau_j \, \mathrm{cl}V(x) \cap V(F) = \emptyset$. Therefore, $\tau_j \, \mathrm{cl}V(x) \subset X \setminus V(F) \subset X \setminus F = U(x)$.

Conversely, if the condition be satisfied, $x \in Y$, $F \in co\tau_i$ and $x \in F$. Then $x \in U(x) = X \setminus F \in \tau_i$ and by condition, there is $V(x) \in \tau_i'$ such that $\tau_j\, clV(x) \subset U(x)$. Hence, we have

$$F = X \setminus U(x) \subset X \setminus \tau_j\, clV(x) = V(F) \in \tau_j \qquad \ldots$$

and $V(x) \cap V(F) = \emptyset$.

(iii) By using (1.2.3) of Definition (1.2.5), taking into account that $x \in Y$, the P_3 (iii) following.

Hence, Y is (i, j)-superregular in X if and only if X is (i, j)- regular at each point of Y .

We know that a TsS (Y, τ') of a TS (X, τ) is WS-regular (WS superregular) in X if for $x \in Y$, $F \in co\tau$ and $x \,\overline{\in}\, F$ there are disjoint sets $U \in \tau'$, $V \in \tau$ such that $x \in U$ and $F \cap Y \subset V$ $(F \subset V)$.

Definition (i- j) regular

Let (Y, τ_1', τ_2') be a BsS of a BS (X, τ_1, τ_2). The point-wise as follow :

i is (i, j)-regular in X from inside if every i-closed in X BsS of Y is (i, j)- regular.

ii Y is (i, j)-internally regular in X if for $x \in Y$, $F \in co\tau_i$, $F \subset Y$ and $x \,\overline{\in}\, F$ there are disjoint sets $U \in \tau_i$, $V \in \tau_j$ such that $x \in U$ and $F \subset V$.

iii X is (i, j)-regular on Y if for each set $F \in co\tau_i$, i- concentrated on Y $(F \subset \tau_i\, cl(F \cap Y))$, and each point $x \in Y \setminus F$ there are disjoint sets $U \in \tau_i$, $V \in \tau_j$ such that $x \in U$ and $F \subset V$.

iv X is (i, j)-strongly regular on Y if for each $F \in co\tau_i$, $i$concentrated on Y, and each point $x \in X \setminus F$ there are disjoint sets $U \in \tau_i$, $V \in \tau_j$ such that $x \in U$ and $F \subset V$.

Theorem

Let $(\boldsymbol{Y}, \tau_1', \tau_2')$ be a BsS of a BS (X, τ_1, τ_2). *Then*

i. *If* Y *is* (i, j)*-regular, then* Y *is* (i, j)*-regular from inside in* X *and so, in every larger* BS.

ii. *If* Y *is* (i, j)*-internally regular in* X*, then* Y *is* (i, j)*-regular in* X *from inside.*

Proof

(i) The proof condition (i) is trivial since if Y is (i, j)-regular, then any BsS of Y is also (i, j)- regular.

(ii) Let $F \in co\tau_i$, $F \subset Y$ and let us prove that F, τ''_1, τ''_2 is (i, j)-regular. If $x \in F$, $\Phi \in co\tau_i''$, $x \in \Phi$, then $x \in Y$ and $\Phi \in co\tau_i$ as $F \in co\tau_i$. Since $x \in Y$, $\Phi \subset Y$, $\Phi \in co\tau_i$, $x \in \Phi$ and Y is (i, j) internally regular in X, there are $U' \in \tau_i$, $V' \in \tau_j$ such that $x \in U'$,

$\Phi \subset V'$ and $U' \cap V' = \emptyset$. Let $U = U' \cap F$, $V = V' \cap F$. Then $U \in \tau_i''$, $V \in \tau_j''$, $x \in U$, $\Phi \subset U$ and $U \cap V = \emptyset$. Thus (F, τ_1'', τ_2'') is (i, j)-regular.

We find that a real-valued function $f : (X, \tau_1, \tau_2) \rightarrow (I, \omega')$ is Y -(i, j)-l.u.s.c. if it is (i, j)-l.u.s.c. at each point $y \in Y$, where (Y, τ_1', τ_2') is a BsS of a BS (X, τ_1, τ_2).

Definition (i-j) almost completely regular

Let (Y, τ_1', τ_2') be a BsS of a BS (X, τ_1, τ_2). Then

i. Y is (i,j)-almost completely regular in X if for each point $x \in Y$ and each F $\in$ co τ'_i, $x \,\overline{\in}$ F set , there is a Y -(j, i)-l.u.s.c. function f : $(X, \tau_1, \tau_2) \rightarrow (I, \omega)$ such that $f(x)=0$ and $f(F) \subset \{1\}$.

ii. Y is (i, j)-completely regular in X if for each point $x \in Y$ and each set $F \in co\tau_i$, $x \bar{\in} F$, there is a (j, i)-l.u.s.c. function $f : (X, \tau_1, \tau_2) \to (I, \omega)$ such that $f(x) = 0$ and $f(F \cap Y) \subset \{1\}$.

iii. Y is (i, j)-strongly completely regular in X if for each point $x \in Y$ and each set $F \in co\tau_i$, $x \bar{\in} F$, there is a (j, i)-l.u.s.c. function $f : (X, \tau_1, \tau_2) \to (I, \omega)$ such that $f(x)=0$ and $f(F)\subset\{1\}$.

Therefore, Y is (i, j)-strongly completely regular in $X \Rightarrow Y$ is (i, j) completely regular in $X \Rightarrow Y$ is (i, j)-almost completely regular in X.

Theorem

A BsS (Y, τ_1', τ_2') *of a* BS (X, τ_1, τ_2) *is* (i, j)- almost completely regular in X if and only if for each point x ∈ Y and any set $U \in \tau_i$, $x \in U$ there *is a* Y -(i, j)-*l.u.s.c.* function $g : (X, \tau_1, \tau_2) \to (I, \omega)$ such that $g(x) = 1$ and $g(X \setminus U) \subset\{0\}$.

Proof

If $P = X \setminus U$, then $P \in co\tau_i$ and $F = P \cap Y \in co\tau_i'$. Since $x \bar{\in} F$ and Y is (i, j)-almost completely regular, there is a Y -(j, i)- l.u.s.c. function f : $(X, \tau_1, \tau_2) \to (I, \omega)$ such that $f(x) = 0$ and $f(F) \subset\{1\}$. If $g' = 1-f$, then g' : $(X, \tau_1, \tau_2) \to (I, \omega)$ is Y -(i, j)-l.u.s.c., $g'(x) = 1$ and $g'(F) \subset\{0\}$. Let $g(x) = g'(x)$ at each point $x \in U$ and $g(x) = 0$ at each point $x \in P = X \setminus U$. Then $g : (X, \tau_1, \tau_2) \to (I, \omega)$ is Y -(i, j)- l.u.s.c., $g(x) = 1$ and $g(X \setminus U) \subset\{0\}$.

Conversely, if the condition be satisfied and $x \in Y$, $F \in co\tau_i'$, $x \bar{\in} F$. Then $x \in \tau_i$ cl$F = P \in co\tau_i$ and so $x \in U = X \setminus P$. Hence, by using the given condition, there is a Y -(i,j)-l.u.s.c. function $f : (X, \tau_1, \tau_2) \to (I, \omega)$ such that $f(x) = 1$ and $f(X \setminus U) \subset\{0\}$. Which implies $g = 1 - f$ is Y -(j, i)l.u.s.c., $g(x) = 0$ and

$g(F) \subset g(P) = 1 - f(P) = 1 - f(X \setminus U) = \{1\}, \quad \ldots$

that is, Y is (i, j)-almost completely regular in X. Hence, the theorem is proved.

Definition (p-quasi normal)

Let (Y, τ_1', τ_2') be a BsS of a BS (X, τ_1, τ_2). Then

i. Y is p-quasi normal in X if for each pair of disjoint sets $A \in co\tau_1$, $B \in co\tau_2$ there are disjoint sets $U \in \tau'_2$, $V \in \tau'_1$ such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

ii. Y is p-normal in X if for each pair of disjoint sets $A \in co\tau_1$, $B \in co\tau_2$ there are disjoint sets $U \in \tau_2$, $V \in \tau_1$ such that $A \cap Y \subset U$ and $B \cap Y \subset V$.

iii. Y is p-strongly normal in X if for each pair of disjoint sets $A \in co\ \tau'_1$, $B \in co\ \tau'_2$ there are disjoint sets $U \in \tau_2$, $V \in \tau_1$ such that $A \subset U$ and $B \subset V$.

iv. Y is (i, j)-supernormal in X if for each pair of disjoint sets $A \in co\tau_i'$, $B \in co\tau_j$ there are disjoint sets $U \in \tau_j$, $V \in \tau_i$ such that $A \subset U$ and $B \subset V$.

v. Y is (i, j)-WS-normal in X if for each pair of disjoint sets $A \in co\tau_i'$, $B \in co\tau_j$ there are disjoint sets $U \in \tau_j'$, $V \in \tau_i$ such that $A \subset U$ and $B \cap Y \subset V$.

vi. Y is (i, j)-WS-supernormal in X if for each pair of disjoint sets $A \in co\tau_i'$, $B \in co\tau_j$ there are disjoint sets $U \in \tau_j'$, $V \in \tau_i$ such that $A \subset U$ and $B \subset V$.

In general, the p-normality of (Y, τ_1', τ_2') in (X, τ_1, τ_2) does not imply the p-normality of (Y, τ_1', τ_2'). Nevertheless, if (Y, τ_1', τ_2') is p-normal in a R-

p-T_1 BS (X, τ_1, τ_2), then (Y, τ_1', τ_2') is p-regular. Indeed, let $x \in Y$, F ∈ co τ'_i and x $\overline{\in}$ F. Then $x \in \tau_i$ cl$F$ and by condition there are disjoint sets $U(x) \in \tau_i$ and $U(\tau_i \text{ cl}F) \in \tau_j$. Clearly, $U'(x) = U(x) \cap Y \in \tau'_i$ and $U'(F) = U(\tau_i \text{ cl}F) \cap Y \in \tau_j'$ are also disjoint, and hence, (Y, τ_1', τ_2') is *p*-regular.

Theorem

Let (Y, τ_1', τ_2') *be a* BsS *of a* BS (X, τ_1, τ_2). Then

i Y is p-strongly normal in X if and only if for each set F ∈ co τ'_1 F ∈ co τ'_2 and any neighbourhood $U'(F) \in \tau'_2$ ($U'(F) \in \tau'_1$) there is a neighborhood $V(F) \in \tau_2$ ($V(F) \in \tau_1$) *such that* $\tau_1 \text{ cl}V(F) \cap Y \subset U'(F)$ ($\tau_2 \text{ cl}V(F) \cap Y \subset U'(F)$).

ii *Y is* (i, j)-WS-supernormal in X if and only if for each set $F \in$ coτ_i' and any neighborhood $U(F) \in \tau_j$ there is a neighborhood $V(F) \in \tau_j'$ such that $\tau_i \text{ cl}V(F) \subset U(F)$.

iii *Y is* (i,j)-supernormal in X if and only if for each set $F \in$ coτ_i' and any neighborhood $U(F) \in \tau_j$ there is a neighborhood $V(F) \in \tau_j$ such that $\tau_i \text{ cl}V(F) \subset U(F)$.

Proof

(i) Let, first, Y be *p*-strongly normal in X, F ∈ co τ'_1 and $U'(F) \in \tau'_2$. Then $\Phi = (X \setminus U(F)) \cap Y$, where $U(F) \in \tau_i$, $U(F) \cap Y = U'(F)$, $\phi \in$ co τ'_2 and F ∩, $\phi \in$ co τ'_2 and $F \cap \phi = \emptyset$. Hence there are $V(F) \in \tau_2$, $V(\Phi) \in \tau_1$ such that $\tau_1 \text{ cl}V(F) \cap V(\Phi) = \emptyset$ and so $V(F) \subset \tau_1 \text{ cl}V(F) \subset X \setminus V(\Phi) \subset X \setminus \Phi = U(F) \cup (X \setminus Y)$.

Therefore, $\tau_1 \text{ cl}V(F) \cap Y \subset U'(F)$.

Conversely, let the condition be satisfied, A $\in$ co τ'_1, B $\in$ co τ'_2 and $A \cap B = \emptyset$. Then A $\subset$ Y \ B = U′(A) $\in \tau'_2$ and by condition there is

$V(A) \in \tau_2$ such that $\tau_1\, \mathrm{cl}V(A) \cap Y \subset U'(A)$. Hence $X \setminus U'(A) \subset X \setminus (\tau_1\, \mathrm{cl}V(A) \cap Y) = (X \setminus \tau_1\, \mathrm{cl}V(A)) \cup (X \setminus Y)$ and so $(X \setminus U'(A)) \cap Y \subset X \setminus \tau_1\, \mathrm{cl}V(A)$, i.e., $B = Y \setminus U'(A) \subset (X \setminus \tau_1\, \mathrm{cl}V(A)) = V(B) \in \tau_1$ and $V(A) \cap V(B) = \emptyset$.

(ii) Let, first, Y be (i, j)-WS-supernormal in X, $A \in \mathrm{co}\tau_i'$ and $U(A) \in \tau_j$. Then $A \cap B = \emptyset$, where $B = X \setminus U(A) \in \mathrm{co}\tau_j$. Hence there are $V(A) \in \tau_j'$, $V(B) \in \tau_i$ such that $\tau_i\, \mathrm{cl}V(A) \cap V(B) = \emptyset$ and thus

$V(A) \subset \tau_i\, \mathrm{cl}V(A) \subset X \setminus V(B) \subset X \setminus B = U(A)$ ...(3.4)

Conversely, let the condition be satisfied, A $\in$ co τ'_i, B $\in$ co τ'_j and $A \cap B = \emptyset$. Then $A \subset U(A) = X \setminus B \in \tau_j$. Hence, by condition, there is V(A) $\in \tau'_j$ such that $\tau_i\, \mathrm{cl}V(A) \subset U(A)$. Therefore,

$B = X \setminus U(A) \subset X \setminus \tau_i\, \mathrm{cl}V(A) = V(B) \in \tau_i$ and $V(A) \cap V(B) = \emptyset$.

Now, the topological version of (i, j)-strong normality, and the topological versions of (i, j) WS-supernormality and (i, j)-supernormality. Hence, theorem is proved.

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